International Journal of Theoretical Physics, Vol. 10, No. 3 (1974), pp. 189–196

Another Form of Einstein's Addition Law of Velocities

P. KALIAPPAN

Department of Mathematics, Nallamuthu Gounder Mahalingam College, Pollachi 642001, Tamil Nadu, India

Received: 30 *September* 1973

Abstract

The velocity of a moving particle is treated in a different way. Accordingly a new form for Einstein's addition law of velocities is given. Lorentz transformations are found from this aspect. The possibility of another way of recognition of the correlation between mass and energy is explained.

1. *Introduction*

In modern views on space, time and motion the velocity of light is not affected by the motion of its source. This means that if we add any other velocity to the velocity of light we again get the same original velocity of light. To make sure of this fact Einstein has given the law of addition of two velocities, u and v, as

$$\frac{u+v}{1+\dfrac{uv}{c^2}} \tag{1.1}$$

where c is the velocity of light. If we add two velocities, no matter how close they are to the velocity of light, the resulting velocity will never exceed c. For example let us put $u = {\cdot}7c$, $v = {\cdot}5c$.

Then the resulting velocity

$$= \frac{{\cdot}7c + {\cdot}5c}{1 + \dfrac{({\cdot}7c)({\cdot}5c)}{c^2}}$$

$$= \frac{1{\cdot}2c}{1{\cdot}35}$$

$$= \frac{120}{135}c < c$$

Even if we take both u, $v > c$, then the resulting velocity will be less than c. For example, let us take $u = 3c$, $v = 5c$, then the resulting velocity will be equal to

$$\frac{3c + 5c}{1 + \frac{(3c)(5c)}{c^2}} = \frac{8}{16}c < c$$

We can also see that the addition of c with any other velocity will give only c. Let us take $u = c$ and let v have any value. The resulting velocity will be equal to

$$\frac{c + v}{1 + \frac{cv}{c^2}} = c\frac{c + v}{c + v} = c$$

Now let us give another law for the addition of two velocities as

$$\frac{uv + c^2}{u + v} \tag{1.2}$$

If we add any two velocities, no matter how close they are to the velocity of light, the resulting velocity will never be less than c. For example, let us take $u = \cdot 7c$, $v = \cdot 5c$, the resulting velocity will then be

$$\frac{(\cdot 7c)(\cdot 5c) + c^2}{(\cdot 7c + \cdot 5c)}$$

$$= \frac{1\cdot 35}{1\cdot 2}c$$

$$= \frac{135}{120}c > c$$

On the other hand, let us take u, $v > c$ as $u = 3c$, $v = 5c$, the resulting velocity will be

$$\frac{(3c)(5c) + c^2}{3c + 5c}$$

$$= \frac{16c^2}{8c} = \frac{16}{8}c > c$$

We shall also show that the addition of c with any other velocity will give only c. For example let us take $u = c$ and let v have any value. The resulting velocity will be equal to

$$\frac{c(v) + c^2}{c + v} = \frac{c(v + c)}{(c + v)} = c$$

Thus we see that the addition laws (1.1) and (1.2) have parallel equivalences. The existence of law (1.2) leads us to define the velocity of a moving particle in a different manner.

2. *Another Definition for the Velocity of a Particle*

Let us consider a particle which has changed its position from A to B. Now the particle is associated with a velocity which is just a rate. Usually we express the velocity as the rate of displacement with time. It is also possible to express the velocity as the rate of change of time with respect to space. For example if a particle describes 5 cm in space and if it takes 10 sec for this description, we say that the velocity of the particle is $\frac{5}{10}$ cm/sec. In a similar way we can define the velocity as $\frac{10}{5}$ sec/cm. The time taken by the particle in 1 cm is considered as the velocity of the particle. To allow this velocity we have to make time play the role of space and space play the role of time. For clarity let us denote the velocity $\frac{5}{10}$ cm/sec as the first kind of velocity and $\frac{10}{5}$ sec/cm as the second kind of velocity.

Now let us turn our attention to the velocity of light. We usually express velocity of light as 3×10^{10} cm/sec. This can as well be expressed as $1/(3 \times 10^{10})$ sec/cm. That is, the light takes time $1/(3 \times 10^{10})$ sec in moving through 1 cm. 3×10^{10} cm/sec is the first kind of velocity of light and $1/(3 \times 10^{10})$ sec/cm is the second kind of velocity of light. In the special theory of relativity we are considering the first type of velocities. Here we note:

The velocity of light is independent of the velocity of its source.

The addition of two velocities will never exceed the velocity of light.

The distance travelled in 1 sec by light is greater. than the distance travelled by any moving particle in 1 sec. Correspondingly Einstein's addition law of velocities takes the form.

$$\frac{u+v}{1+\dfrac{uv}{c^2}}$$

Let us consider the second type of velocities. Even then the velocity of light must be independent of the velocity of its source. But the addition of two velocities will never be less than the velocity of light. Also the time taken by light to travel a distance of 1 cm is less than the time taken by any moving particle to travel a distance of 1 cm.

Correspondingly the law of addition of velocities may be taken as $(uv + c^2)/(u + v)$, where u, v, c are the second type velocities. We shall derive this law in Section 4.

3. *Lorentz Transformations*

Let us consider the velocities of the second type and obtain the corresponding Lorentz transformations. It must be carefully noted that the distance between two positions of a moving particle will be in time. It must be expressed in time units. For example let a particle move with velocity 10 sec/cm. Let us consider two positions, A and B, of the particle. Suppose the distance $AB = 2$ cm. Hereafter we must consider this 2 cm as the space interval between A and B and the distance AB described by the particle may be taken as 20 sec. That is, the particle has travelled through a distance of 20 sec in the interval of space 2 cm. *Therefore if a particle with velocity v sec/cm describes a distance x sec then the space interval for this distance* $= x/v$ *cm.*

Usually when we set up a frame with three coordinate axes, the coordinates (x, y, z) of a point will denote the space lengths. But in our work the coordinates (x, y, z) will denote lengths expressed in time. Usually we take t to denote the time variable. Likewise we will use s to denote the space variable.

Now we shall proceed to obtain Lorentz transformations. Let us consider two inertial coordinate systems T and T'. T' moves relatively to T at the constant rate v (second kind of velocity) along the x-axis. The points of origin of T and T' coincide at the T-space $s = 0$. The x'-axis is parallel to the x-axis and, in fact, coincides with it. Let us assume that a wave front starts from the point of origin of T, which coincides with the point of origin of T' at that moment. The speed of the propagation of the wave is the same in all directions and it is equal to c (second kind of velocity) in terms of either set of coordinate axes.

Usually the progress of the wave is described by either of the two equations

$$x^2 + y^2 + z^2 = c^2 t^2$$

$$x'^2 + y'^2 + z'^2 = c^2 t'^2$$

The transformation equations are taken as

$$x' = \alpha(x - vt), \qquad y = y', \qquad z = z', \qquad t' = \beta t + \lambda x$$

Here x, y, z denote space lengths and t, t' are the time variables. But in our work x, y, z denote time lengths and let us choose s, s' as the space variables. Then the progress of the wave is described by either of the two equations

$$\frac{x^2}{c^2} + \frac{y^2}{c^2} + \frac{z^2}{c^2} = s^2 \tag{3.1}$$

$$\frac{x'^2}{c^2} + \frac{y'^2}{c^2} + \frac{z'^2}{c^2} = s'^2 \tag{3.2}$$

The transformation equation will be

$$s' = \alpha \left(s - \frac{x}{v}\right) \tag{3.3}$$

$$x' = \beta x + \lambda s \tag{3.4}$$

$$y' = y, \qquad z' = z \tag{3.5}$$

Applying (3.3), (3.4) and (3.5) in (3.2) we get

$$\frac{(\beta x + \lambda s)^2}{c^2} + \frac{y^2}{c^2} + \frac{z^2}{c^2} = \alpha^2 \left(s - \frac{x}{v}\right)^2$$

Rearranging the terms we get

$$\left(\frac{\beta^2}{c^2} - \frac{\alpha^2}{v^2}\right) x^2 + \frac{y^2}{c^2} + \frac{z^2}{c^2} + 2\left(\frac{\beta\lambda}{c^2} + \frac{\alpha^2}{v}\right) xs = \left(\alpha^2 - \frac{\lambda^2}{c^2}\right) s^2 \tag{3.6}$$

Equation (3.6) will go over into (3.1) if and only if the coefficients of x^2, s^2 are the same in both equations (3.1) and (3.6) and if the coefficient of xs in equation (3.6) vanishes. Therefore

$$\frac{\beta^2}{c^2} - \frac{\alpha^2}{v^2} = \frac{1}{c^2} \tag{3.7}$$

$$\alpha^2 - \frac{\lambda^2}{c^2} = 1 \tag{3.8}$$

$$\frac{\beta\lambda}{c^2} + \frac{\alpha^2}{v} = 0 \tag{3.9}$$

We solve these three equations for the three unknowns α, β, λ as follows.

From (3.9),

$$\beta = -\frac{\alpha^2 c^2}{v\lambda} \tag{3.10}$$

Inserting this in (3.7), we obtain

$$\frac{\alpha^4 c^4}{v^2 \lambda^2 c^2} - \frac{\alpha^2}{v^2} = \frac{1}{c^2}$$

$$\frac{\alpha^4 c^2}{v^2 \lambda^2} - \frac{\alpha^2}{v^2} = \frac{1}{c^2}$$

$$\alpha^4 c^2 - \alpha^2 \lambda^2 = \frac{v^2 \lambda^2}{c^2} \tag{3.11}$$

From (3.8),

$$\lambda^2 = c^2(\alpha^2 - 1)$$

Inserting this in (3.11),

$$\alpha^4 c^2 - \alpha^2 c^2(\alpha^2 - 1) = \frac{v^2}{c^2} c^2(\alpha^2 - 1)$$

$$\alpha^2 c^2 = v^2\alpha^2 - v^2$$

$$\alpha^2 = \frac{v^2}{v^2 - c^2} = \frac{1}{1 - \frac{c^2}{v^2}} \tag{3.12}$$

By putting this value in (3.7) we will get

$$\beta^2 = \frac{1}{1 - \frac{c^2}{v^2}}$$

From (3.10),

$$\lambda = -\frac{\alpha^2 c^2}{v\beta} = -\frac{c^2}{v}\alpha$$

So we obtain

$$\left.\begin{aligned} \alpha = \beta &= \frac{1}{\sqrt{\left(1 - \frac{c^2}{v^2}\right)}} \\ \text{and} \quad \lambda &= -\frac{c^2}{v}\frac{1}{\sqrt{\left(1 - \frac{c^2}{v^2}\right)}} \end{aligned}\right\} \tag{3.13}$$

By substituting all these values into equations (3.3), (3.4) and (3.5) we get

$$\left.\begin{aligned} s' &= \frac{1}{\sqrt{\left(1 - \frac{c^2}{v^2}\right)}}\left(s - \frac{x}{v}\right) \\ x' &= \frac{1}{\sqrt{\left(1 - \frac{c^2}{v^2}\right)}}\left(x - \frac{c^2}{v}s\right) \\ y' &= y,\ z' = z \end{aligned}\right\} \tag{3.14}$$

The equations given in (3.14) and (3.15) are the Lorentz transformations for the second kind of velocities.

4. *Relativistic Law of Addition of Velocities*

Let us study the superposition of two (or more) Lorentz transformations. We shall introduce three frames of reference, T, T', T''. T' has the velocity v relative to T and T'' has the velocity u relative to T'. Let us find the transformation equations connecting T'' and T.

Let us start with the equations

$$\left.\begin{aligned} x' &= \frac{1}{\sqrt{\left(1-\dfrac{c^2}{v^2}\right)}}\left(x-\frac{c^2}{v}s\right) \qquad y'=y \\ s' &= \frac{1}{\sqrt{\left(1-\dfrac{c^2}{v^2}\right)}}\left(s-\frac{x}{v}\right) \qquad z'=z \end{aligned}\right\} \tag{4.1}$$

$$\left.\begin{aligned} x'' &= \frac{1}{\sqrt{\left(1-\dfrac{c^2}{u^2}\right)}}\left(x'-\frac{c^2}{u}s'\right) \qquad y''=y' \\ s'' &= \frac{1}{\sqrt{\left(1-\dfrac{c^2}{u^2}\right)}}\left(s'-\frac{x'}{u}\right) \qquad z''=z' \end{aligned}\right\} \tag{4.2}$$

Let us substitute the first set of equations into the second set. Straightforward calculation will give

$$\left.\begin{aligned} x'' &= \frac{1}{\sqrt{\left(1-\dfrac{c^2}{w^2}\right)}}\left(x-\frac{c^2}{w}s\right) \\ s'' &= \frac{1}{\sqrt{\left(1-\dfrac{c^2}{w^2}\right)}}\left(s-\frac{x}{w}\right) \\ y'' &= y, \qquad z''=z \end{aligned}\right\} \tag{4.3}$$

with

$$w=\frac{uv+c^2}{u+v} \tag{4.4}$$

Thus the Lorentz transformations, carried out one after another, are equivalent to one Lorentz transformation. Equation (4.4) can be interpreted in the following way. Let a body have the velocity u with respect to T' where T' has the velocity v with respect to T. Then the velocity w of the body with respect to T is given by

$$w = \frac{uv + c^2}{u + v}$$

and so this equation may be regarded as the transformation law for velocities (in the x-direction).

Further, equation (4.4) can be written in the form

$$w = c \frac{1}{1 - \dfrac{(u - c)(v - c)}{uv + c^2}}$$

In this form it is obvious that w cannot become equal or less than c as long as u and v are greater than c. Thus we see that it is impossible to combine several Lorentz transformations in one involving a relative velocity less than c.

5. *Mass and Energy*

Now let us see how the first and second velocities of light connect the mass and energy of a particle. If c denotes the value of the first type of speed of light, then $1/c$ will denote the value of the second type of velocity of light. Let us take the product mc where m is the mass (relativistic or rest) of a particle.

Then $mc = (mc^2)(1/c)$. Thus we see that the product of m with c is equal to the product of mc^2 with $1/c$. But we know that mc^2 represents the energy E of the particle. Thus the equivalence of mass and energy may also be established in the form

(mass)(value of the first type of speed of light)

= energy (value of the second type of speed of light)

Acknowledgements

The author is deeply indebted to the Principal, Professor M. Alkondan, Vice-Principal, Professor N. Namasivayam, Professor K. A. Gundu Rao, Assistant Professor G. Alagar Ramanujam and the Management of Nallamuthu Gounder Mahalingam College, Pollachi, for their kind help and encouragement.